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CHARGE DISTRIBUTION ON THE SURFACE OF A SPHEROIDAL CAVITY IN A DIELECTRIC SOLID†

K. L. CHOWDHURY and P. G. GLOCKNER

Department of Mechanical Engineering, The University of Calgary, Calgary, Alberta, Canada

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Abstract—Using the method of associated matrices, Galerkin-type representations are constructed for the polarization vector $\bar{P}$ and the scalar potential ϕ in a dielectric solid. The representations are used to determine $(\bar{P}, \phi)$ due to a point charge in an infinite dielectric solid. Expressions for $(\bar{P}, \phi)$ are constructed for an ideal dipole, quadrupole and an octapole. By distribution of these multipole singularities with suitable densities along the line segment joining the focii of the spheroid an exact expression for the dielectric displacement caused by a charge distribution over its surface is determined using the singularity method.

INTRODUCTION

Recent years have witnessed application of the singularity method to the construction of a large number of exact solutions to displacement-type boundary value problems for an infinite isotropic medium with spheriodal cavities[1] and various types of Stokes flows over axisymmetric bodies in hydromechanics[2]. The method consists of deriving Kelvin-type solutions and then obtaining solutions for higher order point singularities, known as doublets, centres of dilatation and rotation, stokeslets, stresslets, rotlets, and distributing these singularities, with appropriate densities, along a finite segment of the axis of symmetry of the body to obtain explicit results through integral representation for the unknown displacement. The method has been extended to obtain exact closed form solutions in other areas of mathematical physics, such as magnetostatics, potential theory and scattering of low frequency electromagnetic and acoustic waves[3]. Body shapes treated include spheres, prolate and oblate spheroids, dumbbells and forms generated by a surface of revolution. Compared to the boundary value method, where one encounters overwhelmingly complex analytical difficulties, the singularity technique is simple, elegant and allows exact solutions to boundary value problems to be obtained with ease and in a straightforward logical manner.

In the present work, the application of the singularity method is extended to dielectric materials[4]. Kelvin-type solutions for the polarization vector $\bar{P}$ and scalar potential field ϕ are constructed for a point charge in an infinite dielectric solid using the method of associated matrices[5] and rederiving the solution given in[4]. Point singularities, such as ideal doublets, quadrupoles and octapoles with one, two and three directors, respectively, are generated from the basic point charge solution. Solutions are constructed for both the classical theory of dielectrics as well as for Mindlin's equations which take into account the contribution of the polarization gradient to the stored energy function. Using these multipole point singularities with appropriate densities, an integral representation for the dielectric displacement due to a charge distribution on the surface of the spheroidal cavity in an infinite dielectric material is constructed for a rigid dielectric and an exact solution in terms of elementary functions obtained. The total surface charge is determined by parametrizing the surface of the spheriodal cavity. Detailed explicit evaluations of integrals used in the text are given in appendices.

2. THE BASIC EQUATIONS

Let a homogeneous isotropic centro-symmetric dielectric material occupy a region V whose boundary S separates it from outer vacuum V'. Introduce a rectangular Cartesian coordinate system, x_i.

The basic equations developed in[4] reduce to the equations of equilibrium

$$E_{ij,i} + {}_LE_i - \phi_{,i} = 0 \tag{2.1a}$$

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$$\epsilon_0\phi_{,ii} - P_{i,i} = -\rho_c \text{ in } V \tag{2.1b}$$

$$\phi_{,ii} = 0 \text{ in } V' \tag{2.1c}$$

the constitutive laws

$$_L E_i = -\frac{\partial W^L}{\partial P_i} = -\epsilon_0^{-1}\eta^{-1}P_i \tag{2.2}$$

$$E_{ij} = \frac{\partial W^L}{\partial P_{j,i}} = \beta_1\delta_{ij}P_{k,k} + \beta_2(P_{i,j} + P_{j,i}) + \beta_3(P_{j,i} - P_{i,j}) \tag{2.3}$$

and the boundary conditions

$$n_i E_{ij} = 0 \tag{2.4}$$

$$n_i[P_i - \epsilon_0\|\phi_{,i}\|] = \theta(x) \tag{2.5}$$

where E_{ij}, $P_{i,j}$ denote the components of electric and polarization gradient tensors; $_L E_i$, $\|\phi_{,i}\|$, n_i represent the components of local electric vector, jump in $\phi_{,i}$ across S and the components of unit normal vector to S respectively; η, ϵ_0, $\theta(x)$, ρ_c, respectively, denote the dielectric susceptibility, the permittivity of vacuum, the surface charge and the charge density and β_1, β_2, β_3 are constants associated with the polarization gradient.

The stored energy of deformation and polarization depends on P_i and $P_{i,j}$ and is taken in the form

$$W^L(P_i, P_{i,j}) = \tfrac{1}{2}\alpha_{ij}P_iP_j + \tfrac{1}{2}\beta_{ijkl}P_{j,i}P_{l,k} + \gamma_{ijk}P_iP_{k,j} \tag{2.6}$$

in which for the case of isotropic centro-symmetric dielectric materials, the tensors α_{ij}, β_{ijkl} and γ_{ijk} are defined by

$$\alpha_{ij} = \epsilon_0^{-1}\eta^{-1}\delta_{ij} \tag{2.7a}$$

$$\beta_{ijk} = \beta_1\delta_{ij}\delta_{kl} + \beta_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \beta_3(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{2.7b}$$

$$\gamma_{ijk} = 0 \tag{2.7c}$$

where δ_{ij} is the Kronecker delta.

Eliminating E_{ij} and $_L E_i$ from eqns (2.1a)–(2.1c), (2.4) and (2.5), the following system of equations and boundary conditions is obtained

$$(\beta_2 + \beta_3)\nabla^2\bar{P} + (\beta_1 + \beta_2 - \beta_3)\nabla\nabla\cdot\bar{P} - \epsilon_0^{-1}\eta^{-1}\bar{P} - \bar{\nabla}\phi = 0 \tag{2.8}$$

$$-\epsilon_0\nabla^2\phi + \nabla\cdot\bar{P} = +\rho_c \tag{2.9}$$

$$\beta_1\bar{n}\bar{\nabla}\cdot\bar{P} + 2\beta_2\bar{n}\cdot\bar{\nabla}\bar{P} + (\beta_2 - \beta_3)\bar{n}\times\nabla\times\bar{P} = \bar{0} \tag{2.10}$$

$$\bar{n}\cdot[\bar{P} - \epsilon_o\|\nabla\phi\|] = \theta(x) \tag{2.11}$$

3. GALERKIN'S REPRESENTATION AND POINT CHARGE

Let

$$X_i = \frac{\partial}{\partial x_i}, \quad q = X_1^2 + X_2^2 + X_3^2 \tag{3.1}$$

$$X=\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix},\quad I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad Z=\begin{bmatrix}x_1^2&X_1X_2&X_1X_3\\X_2X_1&X_2^2&X_2X_3\\X_3X_1&X_3X_2&X_3^2\end{bmatrix} \tag{3.2}$$

and let L be a (4×4) matrix associated with eqns (2.8) and (2.9) which are thus rewritten in

$$L\begin{bmatrix}\bar{P}\\ \phi\end{bmatrix}=\begin{bmatrix}\bar{0}\\ +\rho_c\end{bmatrix} \tag{3.3}$$

where

$$L=\begin{bmatrix}\Box_1 I+(\beta_1+\beta_2-\beta_3)Z & -X\\ X' & -\epsilon_0 q\end{bmatrix},\quad \bar{P}=\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix} \tag{3.4}$$

and

$$\Box_1=[-\epsilon_0^{-1}\eta^{-1}+(\beta_2+\beta_3)q]. \tag{3.5}$$

Making use of the results

$$ZX=qX,\quad Z^2=qZ,\quad X'X=q \tag{3.6}$$

$$XX'=Z,\quad X'Z=qX' \tag{3.7}$$

the inverse of the operator L is found to be

$$L^{-1}=\begin{bmatrix}\dfrac{q\Box_2 I+(\Box_1-\Box_2)Z}{q\Box_1\Box_2} & -\dfrac{1}{\epsilon_0 q\Box_2}X\\ \dfrac{1}{\epsilon_0 q\Box_2}X' & -\dfrac{\Box_2+\epsilon_0^{-1}}{\epsilon_0 q\Box_2}\end{bmatrix} \tag{3.8}$$

where

$$\Box_2=(\beta_1+2\beta_2)q-\epsilon_0^{-1}(1+\eta^{-1}). \tag{3.9}$$

From eqns (3.3) and (3.8), one obtains

$$\bar{P}=\epsilon_0[\nabla^2\Box_2+(\Box_1-\Box_2)\nabla\nabla.]\bar{\Phi}-\nabla\psi \tag{3.10}$$

$$\phi=\Box_1\bar{\nabla}\cdot\bar{\Phi}-[\Box_2+\epsilon_0^{-1}]\psi \tag{3.11}$$

where $\bar{\Phi}$ and ψ satisfy the equations

$$\nabla^2\Box_1\Box_2\bar{\Phi}=\bar{0} \tag{3.12}$$

$$\nabla^2\Box_2\psi=+\epsilon_0^{-1}\rho_c. \tag{3.13}$$

Concentrated point charge

Let a concentrated point charge of strength $\rho_c=-4\pi e\,\delta(x)$ be located at the origin of the rectangular coordinate system within an infinite dielectric solid.

Then $\bar{\Phi}=\bar{0}$, and eqn (3.13) reduces to

$$\nabla^2\left(\nabla^2-\frac{1}{l^2}\right)\psi=-4\pi e\,m\,\delta(x) \tag{3.14}$$

where $\delta(x)$ is the Dirac delta function and

$$l^2 = \frac{\beta_1 + 2\beta_2}{\epsilon_0^{-1}(1+\eta^{-1})}, \; m = \frac{1}{l^2}\frac{1}{1+\eta^{-1}}. \tag{3.15}$$

The solution to eqn (3.14) is given by

$$\psi = -e\,l^2 m\left[\frac{1-e^{-R/l}}{R}\right], \quad R^2 = x_1^2 + x_2^2 + x_3^2 \tag{3.16}$$

where we have used

$$\frac{1}{\nabla^2}[-4\pi\delta(x)] = \frac{1}{R} \tag{3.17a}$$

$$\frac{1}{\nabla^2 - \frac{1}{l^2}}[-4\pi\delta(x)] = \frac{e^{-R/l}}{R}. \tag{3.17b}$$

Substituting the expression for ψ from eqn (3.16) into eqns (3.10) and (3.11), one obtains expressions for the polarization vector $\bar{P}$, and the potential ϕ due to a point charge (unipole or 2^0-pole) in the form

$$\bar{P}^e(\bar{x}) = -\frac{\eta e}{1+\eta}\bar{\nabla}\left(\frac{1-e^{-R/l}}{R}\right) \tag{3.18}$$

$$\phi^e(\bar{x}) = \frac{\epsilon_0^{-1}e}{1+\eta}\left(\frac{1+\eta\, e^{-R/l}}{R}\right). \tag{3.19}$$

The dielectric displacement is found from eqns (3.18) and (3.19) and is given by

$$\bar{D}^e(\bar{x}) = \bar{P}^e - \epsilon_0\bar{\nabla}\phi = -e\bar{\nabla}\left(\frac{1}{R}\right) \tag{3.20}$$

which is seen to be independent of the dielectric constants. The expression for ϕ derived here agrees with that obtained by Mindlin[4].

4. MULTIPOLES

In this section, expressions are derived for the potential field, polarization and dielectric vectors for a non-ideal and an ideal dipole, quadrupole and octapole.

Dipole (2-pole)

Consider the point charges $+e$ and $-e$ located at the points with position vector $\bar{r}_0$ and $\bar{r}_0 + l_1\bar{n}_1$ (Appendix A). Define

$$\text{Strength of the dipole} = e l_1 \tag{4.1}$$

$$\begin{aligned}\text{Moment of dipole} = \sum e_i r_i &= -e\bar{r}_0 + e(\bar{r}_0 + l_1\bar{n}_1)\\ &= e l_1 \bar{n}_1\end{aligned} \tag{4.2}$$

$$\text{Potential due to non-ideal dipole} = \sum_{i=1}^{2}\phi_i(P). \tag{4.3}$$

To find the potential due to an ideal dipole, let $e \to \infty$ and $l_1 \to 0$ such that el_1 ($\to \mathcal{S}^d$) remains finite

$$\phi^d(\bar{x}; \bar{n}_1) = +\mathcal{S}^d \frac{\epsilon_0^{-1}}{1+\eta}(\bar{n}_1 \cdot \bar{\nabla})\left(\frac{1+\eta e^{-R/l}}{R}\right) \tag{4.4}$$

$$P^d(\bar{x}; \bar{n}_1) = -\mathcal{S}^d \frac{\eta}{1+\eta}\nabla(\bar{n}_1 \cdot \nabla)\left(\frac{1-e^{-R/l}}{R}\right). \tag{4.5}$$

Notice that the dipole depends on one director $\bar{n}_1$.

Quadrupole (2^2*-pole*)

Consider two dipoles, the first dipole as given above, and the second one obtained from the first by displacing it a distance $l_2\bar{n}_2$ and reversing the position of the charges (see Appendix A) so that $-e$, e, $-e$ and e located at $\bar{r}_0$, $\bar{r}_0 + l_1\bar{n}_1$, $\bar{r}_0 + l_1\bar{n}_1 + l_2 n_2$ and $\bar{r}_0 + l_2\bar{n}_2$, respectively. Define the

$$\text{Strength of the quadrupole} = el_1l_2 \tag{4.6}$$

$$\text{Moment of the quadrupole } \bar{\bar{Q}} = \frac{1}{2!}\sum e_i\bar{r}_i\bar{r}_i \tag{4.7}$$

$$\bar{\bar{Q}} = \frac{el_1l_2}{2!}(\bar{n}_1\bar{n}_2 + \bar{n}_2\bar{n}_1) \tag{4.8}$$

where the charges and their corresponding position vectors are listed in Appendix A.

$$\text{Potential due to non-ideal dipole} = \sum_{i=1}^{4} \phi_i(P). \tag{4.8a}$$

To find the potential due to an ideal quadrupole, let $e \to \infty$, $l_1 \to 0$, $l_2 \to 0$ such that $\lim_{\substack{e\to\infty \\ l_1l_2\to 0}} e\, l_1 l_2 = \mathcal{S}^q$ (finite). The limit of (4.8a) leads to

$$\phi^q(\bar{x}; \bar{n}_1, \bar{n}_2) = +\frac{\epsilon_0^{-1}\mathcal{S}^q}{1+\eta}(\bar{n}_1 \cdot \nabla)(\bar{n}_2 \cdot \nabla)\left[\frac{1+\eta e^{-R/l}}{R}\right]. \tag{4.9}$$

In a similar manner one obtains

$$\bar{P}^q(\bar{x}; \bar{n}_1, \bar{n}_2) = -\frac{\eta}{1+\eta}\mathcal{S}^q\nabla(\bar{n}_1 \cdot \nabla)(\bar{n}_2 \cdot \nabla)\left[\frac{1-e^{-R/l}}{R}\right]. \tag{4.10}$$

Equations (4.9) and (4.10) define the potential and the polarization vector due to a quadrupole located at the origin in an infinite dielectric solid. Notice that the quadrupole depends on two directors, $\bar{n}_1$ and $\bar{n}_2$.

Octapole

An octapole consists of two quadrupoles, the first as given above and the second obtained from the first by displacing it through a distance l_3 and interchanging positions of the charges (Appendix A).

$$\text{Strength of the octapole} = el_1l_2l_3. \tag{4.11}$$

The eight charges $-e$, e, $-e$, e, $-e$, e, $-e$, e are located respectively at $\bar{r}_0$, $\bar{r}_0 + l_1\bar{n}_1$, $\bar{r}_0 + l_1\bar{n}_1 + l_2\bar{n}_2$, $\bar{r}_0 + l_2\bar{n}_2$, $\bar{r}_0 + l_1\bar{n}_1 + l_3\bar{n}_3$, $\bar{r}_0 + l_1\bar{n}_1 + l_3\bar{n}_3 + l_2\bar{n}_2$, $\bar{r}_0 + l_3\bar{n}_3 + l_2\bar{n}_2$, $\bar{r}_0 + l_3\bar{n}_3$.

Moment of the octapole $\bar{\bar{U}} = \frac{1}{3!}\sum_1^8 e_i r_i r_i r_i$ (4.12)

which upon substituting into (4.12) for charges and their position vectors becomes

$$\bar{\bar{U}} = \frac{l_1 l_2 l_3}{3!}(\bar{n}_1\bar{n}_2\bar{n}_3 + \bar{n}_2\bar{n}_3\bar{n}_1 + \bar{n}_3\bar{n}_1\bar{n}_2). \tag{4.13}$$

The potential of an arbitrary point P due to a non-ideal octapole is

$$\sum_{i=1}^{8} \phi_i(P). \tag{4.14}$$

Potential due to an ideal octapole is determined by taking the limit of (4.14) as $e \to \infty$ and l_1, l_2, l_3 each tending to zero, such that $el_1l_2l_3$ remains finite ($= \mathscr{S}^{oct}$) resulting in expressions for the potential and polarization vectors due to an octapole located at the origin in the form

$$\phi^{oct}(\bar{x}; \bar{n}_1, \bar{n}_2, \bar{n}_3) = +\frac{\epsilon_0^{-1}}{1+\eta}\mathscr{S}^{oct}(\bar{n}_1\cdot\bar{\nabla})(\bar{n}_2\cdot\bar{\nabla})(\bar{n}_3\cdot\bar{\nabla}) \times\left(\frac{1+\eta\, e^{-R/l}}{R}\right)$$

$$\bar{P}^{oct}(\bar{x}; \bar{n}_1, \bar{n}_2, \bar{n}_3) = -\mathscr{S}^{oct}\frac{\eta}{1+\eta}\bar{\nabla}(\bar{n}_1\cdot\bar{\nabla})(\bar{n}_2\cdot\bar{\nabla})(\bar{n}_3\cdot\bar{\nabla}) \times\left(\frac{1-e^{-R/l}}{R}\right). \tag{4.15}$$

Notice that the octapole depends on three directors, $\bar{n}_1$, $\bar{n}_2$ and $\bar{n}_3$.

The corresponding expressions for dielectric displacements due to a unipole (2^0-pole), dipole (2-pole), quadrupole (2^2-pole) and octapole (2^3-pole) are obtained from the relation

$$\bar{D} = \bar{P} - \epsilon_0\bar{\nabla}\phi \tag{4.16}$$

and are found to be

$$\bar{D}^e(\bar{x}) = -e\bar{\nabla}\left(\frac{1}{R}\right); \quad \bar{D}^d(\bar{x};\bar{n}_1) = -\mathscr{S}^d\bar{\nabla}(\bar{n}_1\cdot\nabla)\left(\frac{1}{R}\right) \tag{4.17a,b}$$

$$\bar{D}^q(\bar{x}; \bar{n}_1, \bar{n}_2) = -\mathscr{S}^q\bar{\nabla}(\bar{n}_1\cdot\bar{\nabla})(\bar{n}_2\cdot\bar{\nabla})\left(\frac{1}{R}\right) \tag{4.17c}$$

$$\bar{D}^{oct}(\bar{x}; \bar{n}_1, \bar{n}_2, \bar{n}_3) = -\mathscr{S}^{oct}\bar{\nabla}(\bar{n}_1\cdot\bar{\nabla})(\bar{n}_2\cdot\bar{\nabla})(\bar{n}_3\cdot\bar{\nabla})\left(\frac{1}{R}\right) \tag{4.17d}$$

The potential, polarization and dielectric displacement vectors due to higher order singularities can be constructed in a similar manner.

5. SPHEROIDAL CAVITY

Consider an infinite isotropic centro-symmetric dielectric solid with a prolate spheroidal cavity centered at the origin whose surface S is defined by

$$\frac{x^2}{a^2} + \frac{r^2}{b^2} = 1, \quad r^2 = y^2 + z^2, \quad a \geq b. \tag{5.1}$$

The focal length, $2c$, and eccentricity e are related by

$$c = \sqrt{(a^2 - b^2)} = ae, \quad 0 \le e \le 1. \tag{5.2}$$

For a rigid dielectric in which the polarization gradient effects are negligible, one is interested in determining the dielectric displacement field such that it satisfies

(a) Maxwell's equation

$$\bar{\nabla} \cdot \bar{D} = 0 \tag{5.3}$$

(b) Boundary condition

$$n \cdot \bar{D} = \bar{n} \cdot \bar{d} \tag{5.4}$$

where

$$\bar{d} = (d_0 + d_1 x)\bar{e}_x + d_2 \bar{e}_y$$

and

$$\hat{n} = \frac{b/a}{\sqrt{(a^2 - e^2 x^2)}} \left(x\bar{e}_x + \frac{y}{1 - e^2} \bar{e}_y + \frac{z}{1 - e^2} \bar{e}_z \right) \tag{5.5}$$

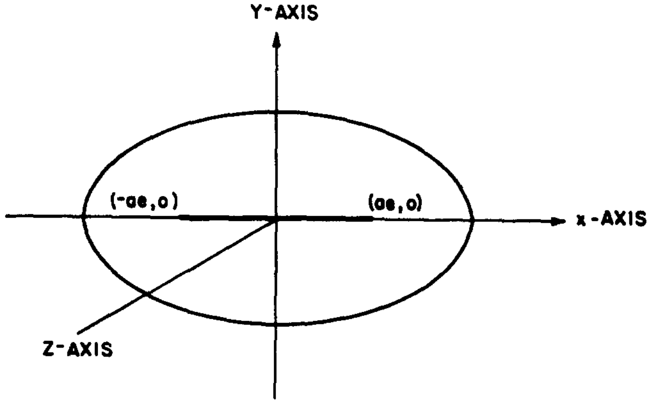
is a unit vector normal to the surface of the cavity and $\bar{e}_x$, $\bar{e}_y$, $\bar{e}_z$ are unit vectors in the direction of the coordinate axes and d_0, d_1, d_2 are arbitrary constants.

(c) and the vanishing of the dielectric displacement at infinity.

Guided by the structure of the solutions derived in[1-3] construct the solution to eqn (5.3) and (5.4) by considering a point charge singularity with constant density, a dipole singularity with $\bar{n}_1 = \bar{e}_x, \bar{e}_y$ and a density $= c^2 - \xi^2$, a quadrupole singularity with $\bar{n}_1 = \bar{n}_2 = \bar{e}_x$ and density $= (c^2 - \xi^2)^2$ distributed along the line segment $(-ae, 0)$ to $(ae, 0)$ on the x-axis between the focii of the spheroid. Assume the solution in the form of an integral representation as

$$\bar{D}(\bar{x}) = -\int_{-c}^{c} [A_1 \bar{D}^e(\bar{x} - \bar{\xi}) + A_2 D^d(\bar{x} - \bar{\xi}; \bar{e}_x)(c^2 - \xi^2)$$

$$+ A_3 D^3(\bar{x} - \bar{\xi}; \bar{e}_y)(c^2 - \xi^2) + A_4 \bar{D}^q(\bar{x} - \bar{\xi}; \bar{e}_x, \bar{e}_x)(c^2 - \xi^2)^2] \mathrm{d}\, \xi \tag{5.6}$$



PROLATE SPHEROID: $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$, $a \ge b$

Fig. 1. Line segment for singularities distribution: $(-ae, 0)$–$(ae, 0)$.

where $\bar{\xi} = \xi\bar{e}_x$ and $A_i (i = 1,2,3,4)$ are arbitrary constants to be determined from the boundary conditions.

Substituting for $D^e(\bar{x}-\bar{\xi},\ D^d(\bar{x}-\bar{\xi};\bar{e}_x),\ D^d(\bar{x}-\bar{\xi};\bar{e}_y)$ and $D^q(\bar{x}-\bar{\xi};\ \bar{e}_x,\ \bar{e}_x)$, eqn (5.6) may be rewritten as

$$\bar{D}(\bar{x}) = \nabla \int_{-c}^{c} \left[A_1 + A_2(c^2-\xi^2)\frac{\partial}{\partial x} + A_3(c^2-\xi^2)^2\frac{\partial^2}{\partial x^2} + A_4(c^2-\xi^2)\frac{\partial}{\partial y}\right]\left(\frac{1}{R_\xi}\right)\mathrm{d}\xi \tag{5.7}$$

where $R = [(x-\xi)^2 + y^2 + z^2]^{1/2}$ and we have assumed that $e = S^d = \mathscr{S}^q = \mathscr{S}^{oct} = 1$.

The boundary condition (5.4) assumes the form

$$\frac{b/a}{\sqrt{(a^2-e^2x^2)}}[d_0 x + d_1 x^2 + d_2(1-e^2)^{-1} y] = \sum_{i=1}^{4} A_i I_i \tag{5.8}$$

where

$$I_1 = (\bar{n}\cdot\bar{\nabla})\left[\int_{-c}^{c}\frac{1}{R_\xi}\mathrm{d}\,\xi\right] \tag{5.9a}$$

$$I_2 = (\bar{n}\cdot\bar{\nabla})\left[\int_{-c}^{c}(c^2-\xi^2)\frac{\partial}{\partial x}\left(\frac{1}{R_\xi}\right)\mathrm{d}\,\xi\right] \tag{5.9b}$$

$$I_3 = (\bar{n}\cdot\bar{\nabla})\left[\int_{-c}^{c}(c^2-\xi^2)^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{R_\xi}\right)\mathrm{d}\,\xi\right] \tag{5.9c}$$

$$I_4 = (\bar{n}\cdot\bar{\nabla})\left[\int_{-c}^{c}(c^2-\xi^2)\frac{\partial}{\partial y}\left(\frac{1}{R_\xi}\right)\mathrm{d}\xi\right]. \tag{5.9d}$$

Detailed evaluation of the integrals I_i (i = 1–4) is given in Appendix C which on the surface of a spheroid become

$$I_1 = -\frac{b/a}{\sqrt{(a^2-e^2x^2)}}\left[\frac{2e}{1-e^2}\right] \tag{5.10a}$$

$$I_2 = +\frac{2b/a}{\sqrt{(a^2-e^2x^2)}}\left[-L+\frac{2e}{1-e^2}\right]x \tag{5.10b}$$

$$I_3 = \frac{b/a}{\sqrt{(a^2-e^2x^2)}}\left[(3L-4e-\frac{2e}{1-e^2}\Big)(-a^2+3x^2)\right] \tag{5.10c}$$

$$I_4 = +\frac{b/a}{\sqrt{(a^2-e^2x^2)}}\left[\frac{1}{1-e^2}\left(+L-4e+\frac{2e}{1-e^2}\right)y\right]. \tag{5.10d}$$

Substituting from (5.10) into (5.8), and comparing coefficients of 1, x, y, x^2, one obtains a linear system of algebraic equations leading to

$$A_1 = -\frac{a^2}{3}\left[\frac{2e}{1-e^2}\right]^{-1} d_1,\quad A_2 = +\frac{d_0}{2}\left[-L+\frac{2e}{1-e^2}\right]^{-1} \tag{5.11a}$$

$$A_3 = \frac{d_1}{3}\left[3L-4e-\frac{2e}{1-e^2}\right]^{-1},\quad A_4 = d_2\left[+L-4e+\frac{2e}{1-e^2}\right]^{-1} \tag{5.11b}$$

thus determining, explicitly, the expression for the dielectric displacement as

$$\bar{D}(\bar{x}) = \nabla[B_{1,0}A_1 - 2B_{1,1}A_2 - y(c^2B_{3,0} - B_{3,2})A_4 - 4(c^2B_{1,0} - 3B_{1,2})A_3]$$
$$= \nabla\left\{[A_1 - 2xA_2 - 2(2c^2 - 6x^2 + 9r^2)A_3 + yA_4]\log\frac{R_2 - \overline{x-c}}{R_1 - \overline{x+c}}\right.$$
$$\left. + 2(R_1 - R_2)[A_2 - 3(3x - c)A_3] - \frac{y}{r^2}R_1R_2\left[\frac{x+c}{R_1} - \frac{x-c}{R_2}\right]A_4\right\} \quad (5.12)$$

where $B_{n,n}$ with its recurrence formulae and evaluations on the surface of the spheroid are given in Appendix B.

Total surface charge

The total charge C on the surface of the spheroid is given by

$$C = b/a \oint\!\!\oint_S \left[\frac{d_0x + d_1x^2 + d_2(1-e^2)^{-1}y}{\sqrt{(a^2 - e^2x^2)}}\right] dS. \quad (5.13)$$

Using a parametric representation for the surface S of a spheroid

$$x = c\xi\eta, \quad y + iz = c\sqrt{((\xi^2 - 1)(1 - \eta^2))}\, e^{i\theta} \quad (5.14)$$
$$-1 \le \eta \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 < \xi < \infty$$

and $\xi = \xi_0$ on S, one finds

$$a = c\xi_0, \quad b = c\sqrt{(\xi_0^2 - 1)}, \quad c = ae, \quad e = \frac{1}{\xi_0} \quad (5.15)$$

$$\bar{r} = c\left[\xi_0\eta, \sqrt{(\xi_0^2 - 1)}\sqrt{(1-\eta^2)}\cos\theta, \sqrt{(\xi_0^2 - 1)}\sqrt{(1-\eta^2}\sin\theta\right] \quad (5.16)$$

$$dS = \left|\frac{\partial\bar{r}}{\partial\eta} \times \frac{\partial\bar{r}}{\partial\theta}\right| d\eta\, d\theta = a^2\sqrt{(1-e^2)}\sqrt{(1-e^2\eta^2)}\, d\eta d\theta \quad (5.17)$$

resulting in an expression for C on the surface of the spheroid as

$$C = ab^2\int_0^{2\pi}\int_{-1}^{+1}\left[d_0\eta + d_1a\eta^2 + \frac{d_2}{\sqrt{(1-e^2)}}\sqrt{(1-\eta^2)}\cos\theta\right] d\eta\, d\theta = \frac{4\pi}{3}a^3(1-e^2)\, d_1. \quad (5.18)$$

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APPENDIX A

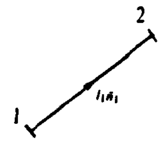
The unipole, dipole, quadrupole and octapole, together with the position vectors of the locations of their charges are listed below.

(*A*) Unipole (2^0-pole) consists of a single point charge.

$$\begin{array}{c}1\\ \downarrow\\ (e, \bar{r}_0)\end{array}$$

(*b*) Dipole (2^1-pole) consists of two point charges.

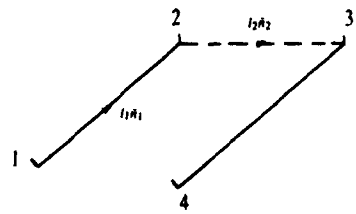
$$\begin{array}{c}1\\ \downarrow\\ (-e, \bar{r}_0)\end{array} \qquad \begin{array}{c}2\\ \downarrow\\ (e, \bar{r}_0 + l_1\bar{n}_1)\end{array}$$



(*c*) Quadrupole (2^2-pole) consists of four point charges.

$$\begin{array}{c}1\\ \downarrow\\ (-e, \bar{r}_0)\end{array} ; \quad \begin{array}{c}2\\ \downarrow\\ (e, \bar{r}_0 + l_1\bar{n}_1)\end{array} ; \quad \begin{array}{c}3\\ \downarrow\\ (-e, \bar{r}_0 + l_1\bar{n}_1 + l_2\bar{n}_2)\end{array}$$

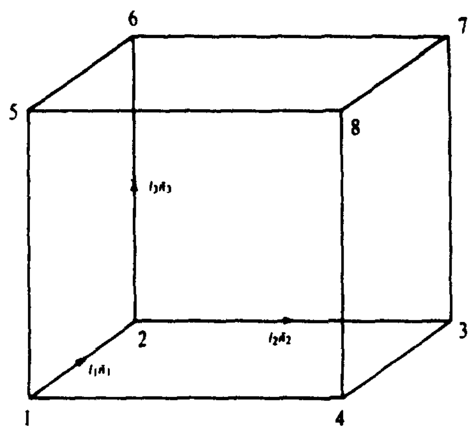
$$\begin{array}{c}4\\ \downarrow\\ (e, \bar{r}_0 + l_2\bar{n}_2)\end{array}$$



(*d*) Octapole (2^3-pole) consists of eight point charges.

$$\begin{array}{c}1\\ \downarrow\\ (-e, \bar{r}_0)\end{array} \quad \begin{array}{c}2\\ \downarrow\\ (e, \bar{r}_0 + l_1\bar{n}_1)\end{array} \quad \begin{array}{c}3\\ \downarrow\\ (-e, \bar{r}_0 + l_1\bar{n}_1 + l_2\bar{n}_2)\end{array} \quad \begin{array}{c}4\\ \downarrow\\ (e, \bar{r}_0 + l_2\bar{n}_2)\end{array} \quad \begin{array}{c}5\\ \downarrow\\ (-e, \bar{r}_0 + l_1\bar{n}_1 + l_3\bar{n}_3)\end{array}$$

$$\begin{array}{c}6\\ \downarrow\\ (e, \bar{r}_0 + l_1\bar{n}_1 + l_2\bar{n}_2 + l_3\bar{n}_3)\end{array} \quad \begin{array}{c}7\\ \downarrow\\ (-e, \bar{r}_0 + l_2\bar{n}_2 + l_3\bar{n}_3)\end{array} \quad \begin{array}{c}8\\ \downarrow\\ (e, \bar{r}_0 + l_3\bar{n}_3)\end{array}$$



APPENDIX B

The form of $B_{m,n}$, its recurrence formula and some commonly used results are given below:

$$B_{m,n} = \int_{-c}^{c} \frac{1}{R_\xi^m} \xi^n \, d\xi, \quad R_\xi^2 = (x-\xi)^2 + r^2. \tag{B1}$$

Recurrence Formula:

$$B_{m,n} = -\frac{c^{n-1}}{m-2}\left(\frac{1}{R_2^{m-2}} + \frac{(-1)^n}{R_1^{m-2}}\right) + \frac{n-1}{m-2} B_{m-2,n-2} + xB_{m,n-1}, \quad n \geq 2. \tag{B2}$$

Differential Relations:

$$\frac{\partial}{\partial x} B_{m,n} = -m[xB_{m+2,n} - B_{m+2,n+1}] \tag{B3}$$

$$\frac{\partial}{\partial r} B_{m,n} = -mr B_{m+2,n}. \tag{B4}$$

Some Evaluations:

$$B_{1,0} = \log \frac{R_2 - \overline{x - c}}{R_1 - \overline{x + c}} \tag{B5}$$

$$B_{1,1} = R_2 - R_1 + xB_{1,0} \tag{B6}$$

$$B_{3,0} = \frac{1}{r^2}\left(\frac{x+c}{R_1} - \frac{x-c}{R_2}\right), \quad B_{31} = \frac{1}{R_1} - \frac{1}{R_2} + xB_{30} \tag{B7}$$

$$B_{5,0} = \frac{1}{3r^2}\left[\frac{x+c}{R_1^3} - \frac{x-c}{R_2^3} + \frac{2}{r^2}\left(\frac{x+c}{R_1} - \frac{x-c}{R}\right)\right] \tag{B8}$$

$$B_{5,1} = \frac{1}{3}\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] + xB_{5,0} \tag{B9}$$

$$B_{5,2} = -\frac{c}{3}\left(\frac{1}{R_2^3} + \frac{1}{R_1^3}\right) + \frac{x}{3}\left(\frac{1}{R_1^3} - \frac{1}{R_2^3}\right) + \frac{1}{3} B_{30} + x^2 B_{50} \tag{B10}$$

$$B_{5,3} = -\frac{c^2}{3}\left(\frac{1}{R_2^3} - \frac{1}{R_1^3}\right) + \frac{2}{3} B_{31} + xB_{52} \tag{B11}$$

$$B_{5,4} = -\frac{c^3}{3}\left(\frac{1}{R_2^3} + \frac{1}{R_1^3}\right) + \frac{c^2 x}{3}\left(\frac{1}{R_1^3} - \frac{1}{R_2^3}\right) + B_{32} + \frac{2x}{3} B_{31} + x^2 B_{52} \tag{B12}$$

where

$$R_1 = [(x+c)^2 + r^2]^{1/2}, \quad R_2 = [(x-c)^2 + r^2]^{1/2} \tag{B13}$$

$$r^2 = y^2 + z^2. \tag{B13a}$$

On the Surface of the Spheroid:

$$r^2 = (1-e^2)(a^2 - x^2), \quad R_1 = a + ex, \quad R_2 = a - ex \tag{B14}$$

$$B_{1,0} = \log \frac{1+e}{1-e} = L \tag{B15}$$

$$B_{3,0} = \frac{2e}{(1-e^2)(a^2 - e^2x^2)}, \quad B_{31} = \frac{2e^3 x}{(1-e^2)(a^2 - e^2 x^2)} \tag{B16}$$

$$a^2 B_{3,0} - xB_{31} = \frac{2e}{1-e^2}, \quad B_{32} = L - \frac{2e}{1-e^2} + \frac{2a^2 e^3}{(1-e^2)(a^2 - e^2 x^2)} \tag{B17}$$

$$c^2 B_{3,0} - B_{32} = -L + \frac{2e}{1-e^2} \tag{B18}$$

$$a^2 B_{3,2} - xB_{3,3} = a^2(L - 2e) - x^2\left(3L - 4e - \frac{2e}{1-e^2}\right) \tag{B19}$$

$$a^2 c^2 B_{30} - xc^2 B_{31} - a^2 B_{32} + xB_{33} = a^2\left(-L + \frac{2e}{1-e^2}\right) + x^2\left(3L - 4e - \frac{2e}{1-e^2}\right) \tag{B20}$$

$$a^2 c^2 B_{50} - xc^2 B_{51} - a^2 B_{52} + xB_{53} = \frac{4e^3}{3(1-e^2)^2} \tag{B21}$$

$$a^2 c^2 B_{51} - xc^2 B_{52} - a^2 B_{53} + xB_{54} = x\left[L - \frac{8e}{3(1-e^2)} + \frac{2e}{3}\frac{1+e^2}{(1-e^2)^2}\right] \tag{B22}$$

APPENDIX C

Evaluation of Integrals I_i ($i = 1$–4) on S.
(a) The integral I_1:

$$I_1 = (\bar{n} \cdot \bar{\nabla}) \int_{-c}^{c} \frac{1}{R_\xi} \mathrm{d}\xi \tag{C1}$$

Making use of the result

$$\hat{n}\cdot[(x-\xi)\bar{e}_x+y\bar{e}_y+z\bar{e}_z]=\frac{b/a}{\sqrt{(a^2-e^2x^2)}}[a^2-x\xi] \tag{C2}$$

$$I_1=-\frac{b/a}{\sqrt{(a^2-e^2x^2)}}[a^2B_{30}-xB_{31}]. \tag{C3}$$

Substituting from (B17) into (C3), one obtains

$$I_1=-\frac{b/a}{\sqrt{(a^2-e^2x^2)}}\left[\frac{2e}{1-e^2}\right]. \tag{C4}$$

(*b*) The integral I_2:

$$I_2=(\hat{n}\cdot\bar{\nabla})\left[\int_{-c}^{c}(c^2-\xi^2)\frac{\partial}{\partial x}\left(\frac{1}{R_\xi}\right)\mathrm{d}\xi\right]. \tag{C5}$$

Since $(\partial/\partial x)\,(1/R_\xi)=-(\partial/\partial\xi)\,(1/R_\xi)$, integrating by parts, we find

$$\begin{aligned}I_2&=-(\hat{n}\cdot\bar{\nabla})\left[(c^2-\xi^2)\frac{1}{R_\xi}\Big|_{\xi=-c}^{\xi=c}+\int_{-c}^{c}\frac{2\xi}{R_\xi}\mathrm{d}\xi\right]\\&=-2(\hat{n}\cdot\bar{\nabla})\int_{-c}^{c}\frac{\xi}{R_\xi}\mathrm{d}\xi=\frac{2b/a}{\sqrt{(a^2-e^2x^2)}}\int_{-c}^{c}\frac{(a^2-\xi x)\xi}{R_\xi^3}\mathrm{d}\xi\\&=\frac{2b/a}{\sqrt{(a^2-e^2x^2)}}[a^2B_{31}-xB_{32}]\end{aligned} \tag{C6}$$

where (C2) was used.

Substituting from (B16,17) into (C6), one finds

$$I_2=-\frac{2b/a}{\sqrt{(a^2-e^2x^2)}}\left[L-\frac{2e}{1-e^2}\right]x. \tag{C7}$$

(*c*) The integral I_3:

$$I_3=(\hat{n}\cdot\bar{\nabla})\int_{-c}^{c}(c^2-\xi^2)\frac{\partial^2}{\partial x^2}\left(\frac{1}{R_\xi}\right)\mathrm{d}\xi. \tag{C8}$$

Integrating by parts twice and using the result (C2), there results

$$I_3=\frac{b/a}{\sqrt{(a^2-e^2x^2)}}\int_{-c}^{c}\frac{(a^2-\xi x)(c^2-3\xi^2)}{R_\xi^3}\mathrm{d}\xi. \tag{C9}$$

Making use of (B19) and (B20), one obtains

$$I_3=\frac{b/a}{\sqrt{(a^2-e^2x^2)}}\left[(3x^2-a^2)\left(3L-4e-\frac{2e}{1-e^2}\right)\right]. \tag{C10}$$

(*d*) The integral I_4:

$$\begin{aligned}I_4&=(\hat{n}\cdot\bar{\nabla})\frac{\partial}{\partial y}\int_{-c}^{c}\left[(c^2-\xi^2)\frac{1}{R_\xi}\right]\mathrm{d}\xi\\&=\frac{-b/a\,y}{\sqrt{(a^2-e^2x^2)}}\left[\frac{1}{1-e^2}(c^2A_{30}-A_{32})-3(a^2c^2B_{50}-xc^2B_{51}-a^2B_{52}+xB_{53})\right]\end{aligned} \tag{C11}$$

which, on S becomes

$$I_4=\frac{-b/a\,y}{\sqrt{(a^2-e^2x^2)}}\left[-\frac{4e^3}{(1-e^2)^2}+\frac{1}{1-e^2}\left(-L+\frac{2e}{1-e^2}\right)\right] \tag{C12}$$

where we have used (B18) and (B21).